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2007 J. Phys. A: Math. Theor. 40 2803

(http://iopscience.iop.org/1751-8121/40/11/015)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **40** (2007) 2803–2818 doi:10.1088/1751-8113/40/11/015

Explicit computation of the spectrum of the deformed Calogero model by Yukawa-like potential

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Received 21 October 2006, in final form 25 January 2007
Published 28 February 2007
Online at stacks.iop.org/JPhysA/40/2803

Abstract
We study the quantum spectrum of the one-dimensional Calogero model deformed by a Yukawa-like potential. Using special features of the UV and IR behaviour of the deformed potential, $\mathbb{Z}_2$ symmetry of classical Hamiltonian as well as the expansion method of the wavefunction, we compute the explicit expression of the discrete energy spectrum $\mathcal{E}_n$ and the corresponding eigenfunctions Υ_n.

PACS numbers: 02.30.Ik, 03.65.Db

1. Introduction

Integrable systems give exact information on energy spectrum [1, 2], and their study has lead to the discovery of many remarkable features of integrable Hamiltonians and their classification [3–10]. However, despite the variety of methods used to approach integrability, there is only a few number of physical systems that are completely solvable and are moreover very special in the sense that the physical situations they describe are very limited. Most of the real physical systems are far away from these models and the determination of their spectrum is not an easy task. There, the main difficulty in getting exact solutions comes essentially from the fact that the underlying wave equations are nonlinear coupled differential equations difficult to solve. To deal with this basic difficulty, one may use the lessons learnt from the study of integrable models stipulating that Hamiltonians may, in general, be classified into three basic sets:

(1) Integrable systems where the spectrum of the Hamiltonian is exactly determined as in the case of a quantum harmonic oscillator, Calogero model and extensions [3, 11].
(2) Quasi-integrable systems [12] where, though not completely determined, the Hamiltonian spectrum obtained by the deformation of exact models is under control.

(3) Remaining other systems with unknown spectrum to which belong most of the real physical situations.

In this paper, we develop an explicit study concerning the determination of the quantum spectrum of a model belonging to the class of integrable systems. This analysis, which will be further developed in the following section, describes a particular deformation of Calogero interaction where the Calogero particles are supposed moreover to be strongly correlated. The strong coupling between the pairs of particles $\{x_i, x_j\}$ is implemented by the adjunction to the usual Calogero Hamiltonian $\mathcal{H}_{\mathrm{cal}}$ an extra short distance interaction described by a Yukawa-like coupling given by

$$V_{\mathrm{Yuk}} = \sum_{i>j} \frac{2\beta}{|x_i - x_j|} \exp\left(-\frac{|x_i - x_j|}{\lambda}\right), \tag{1}$$

where β is a coupling constant and λ is the Debye length [13, 14]. Among our basic results, we find, in case of a system of two particles the following general discrete energy formula:

$$E_n = \left(n + \epsilon + \frac{1}{2}\right)\omega + \frac{2\beta(-1)^{n+1}}{\lambda(n+1)}, \tag{2}$$

with $n = 2m + 1$. Note that the condition n odd captures the quantum implementation of the symmetry of the classical Hamiltonian under the permutation $x_i \leftrightarrow x_j$; for more details see equations (56)–(57). Note also that in the limit $\lambda \to \infty$ or in the case where $\beta = 0$, one recovers the usual Calogero spectrum. Moreover, using the limit $\lambda \to 0$, consistency requires that $\frac{\beta}{\lambda}$ should be a constant which by using dimensional argument should be proportional to ω. Putting these data above we get, for the example $\frac{\beta}{\lambda} = \omega$, the exact result

$$\mathcal{E}_m = \left(2m + \epsilon + \frac{3}{2} + \frac{1}{m+1}\right)\omega, \qquad m = 0, 1, \ldots. \tag{3}$$

This paper is organized as follows. In section 2, we describe the deformed Calogero model by a Yukawa-like interaction modelling strong correlations. In section 3, we develop the explicit computation of the spectrum of this model. In section 4, we make a conclusion, and in section 5, we give two appendices A and B where we give the proofs of propositions used to determine the quantum spectrum of the deformed model.

2. Deformed Calogero system (DCS)

To begin recall that a one-dimensional system of identical particles, having the pairwise inverse square and harmonic interactions [1, 2], known in the literature as the Calogero model, has generated wide interest. This is an exactly solvable model and its generalizations to the periodic case [15] and the spin systems [16] have been found relevant for the description of various physical phenomena such as the universal conductance fluctuations in mesoscopic systems [17], quantum Hall effect [18], wave propagation in stratified fields [19], random matrix theory [15, 20], fractional statistics [21], two-dimensional gravity [22] and gauge theories [23].

In this section, we develop the study of the deformation of Calogero coupling by a Yukawa-like interaction V_{Yuk} introduced in [13]. More precisely, we set up the basic tools towards the explicit computation of the DCS-quantum energy spectrum (2).

First, we motivate integrability of the deformation (1) in terms of boundary conditions of the Calogero model to be considered later. Note that though V_{Yuk} is highly nonlinear, we will show that DCS integrability is possible due to the behaviour of

$$V_{\mathrm{def}}(x) = V_{\mathrm{cal}}(x) + V_{\mathrm{Yuk}}(x) \tag{4}$$

near the half-line boundaries $x \to 0$ and $x \to \infty$. For the proof of this statement, we shall fix our attention below on the case of two particles. Then we solve DCS-Schrödinger equation by using the expansion method of the wavefunction.

2.1. DCS-Hamiltonian model

We start by recalling that the one-dimensional Calogero model describing the quantum dynamics of two interacting particles, of local coordinates x_1 and x_2 and relative position $x = \frac{(x_2 - x_1)}{\sqrt{2}}$, is governed by the following eigenvalue wave equation:

$$\mathcal{H}_{\text{cal}} \Psi_n^{\text{cal}}(x) = E_n^{\text{cal}} \Psi_n^{\text{cal}}(x). \tag{5}$$

The Calogero Hamiltonian operator is

$$\mathcal{H}_{\text{cal}} = \left(-\frac{1}{2} \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{1}{2} \omega^2 x^2 + \frac{g}{2x^2} \right), \tag{6}$$

where, to fix the ideas, we have taken $x > 0$, i.e $x_2 > x_1$; but at the quantum level the symmetry $x \leftrightarrow -x$ of the Hamiltonian should be imposed as a condition on the energy spectrum of the underlying Hilbert space; see equations (2)–(3). For convenience, we shall sometimes drop out the quantum number n on E_n and Ψ_n; it will be re-inserted whenever needed. In above relation, ω is the usual harmonic oscillator frequency and g is the Calogero coupling parameter. As is well known, g has the factorization $g = \epsilon(\epsilon - 1)$, with ϵ is a real modulus ($\epsilon > \frac{-1}{2}$ as required by quantum mechanics) and equations (5)–(6) have exact solutions [1]. Note that the Schrodinger problem for the 1D Calogero model coincides with the Schrodinger problem for the radial part of the spherical oscillator. The latter has been subject to a considerable interest in connection with isotropic confining potentials [24]; for a recent study on spherically confined isotropic harmonic oscillator see [25] and references therein.

In present work, we are interested to study the case where the Calogero particles are moreover strongly correlated. The typical interaction describing this particular behaviour is given by the following Yukawa-like potential [26]:

$$V_{\text{Yuk}}(x) = \frac{2\beta}{x} \exp\left(-\frac{x}{\lambda}\right), \qquad x\sqrt{2} = |x_1 - x_2|, \tag{7}$$

where β is a coupling constant and λ is the Debye length. Note in passing that Yukawa interaction had been used successfully in the phenomenological description of strong force at small energies. Note moreover that the deformation of Calogero interaction by V_{Yuk} has a very remarkable property which turns out to play a central role in the study of the deformation. This property, to be given below, can be viewed as capturing a basic information for dealing with integrable models obtained from deformations of the 1D Calogero model. More precisely, the interaction potential $V_{\text{def}}(x)$ in the DCS Hamiltonian

$$V_{\text{def}}(x) = \frac{g}{2x^2} + V_{\text{Yuk}}(x) \tag{8}$$

tends towards the Calogero potential at both boundary limits $x \to 0$ and $x \to \infty$; we have

$$V_{\text{def}}(x) \underset{x\to 0}{\to} \frac{g}{2x^2}, \tag{9}$$

$$V_{\text{def}}(x) \underset{x\to \infty}{\to} \frac{g}{2x^2}. \tag{10}$$

This property makes DCS Hamiltonian very special in the following sense.

(i) It let understand that $\Psi_{\mathrm{cal}}(x)$ and $\Psi_{\mathrm{def}}(x)$ waves have quite similar behaviour at $x \to 0$ and $x \to \infty$. So, one expects that the DCS wave equation,

$$\left(-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{1}{2}\omega^2 x^2 + V_{\mathrm{def}}(x)\right)\Psi(x) = E\Psi(x), \tag{11}$$

to be integrable. By integrability, we mean that, using properties (9)–(10), we can compute the quantum spectrum $\{E_n, \Psi_n\}$ of the deformed system.

(ii) The total (probability) density $\int_0^\infty \mathrm{d}x\, |\Psi_n(x)|^2$ is finite and involves quite same conditions on coupling constants as for the Calogero model $\left(\epsilon > \frac{-1}{2}, g > \frac{-1}{4}\right)$. The difficulties of the convergence of the integral of $|\Psi_n(x)|^2$ at the boundaries $x = 0$ and $x = \infty$ are more or less same as the usual ones for Calogero waves. Because of equations (9)–(10), the total probability integral may be then split as

$$\int_0^\infty \mathrm{d}x|\Psi(x)|^2 \simeq \int_0^{\xi_1} \mathrm{d}x|\Psi_{\mathrm{cal}}(x)|^2 + \int_{\xi_1}^{\xi_2} \mathrm{d}x|\Psi(x)|^2 + \int_{\xi_2}^{\infty} \mathrm{d}x|\Psi_{\mathrm{cal}}(x)|^2. \tag{12}$$

In this decomposition, ξ_1 is a small positive number and $[0, \xi_1]$ is the region where equation (10) holds ($\Psi_{\mathrm{def}}(x) \sim \Psi_{\mathrm{cal}}(x)$). ξ_2 is a large number and can be thought of as $\frac{1}{\xi_1}$.

Establishing integrability of DCS is the main purpose of the present study; this will be done in steps; but before that let us derive the conditions under which equations (9)–(10) hold.

In the range $x \ll \lambda$ $\left(y = \frac{x}{\lambda} \ll 1\right)$, the two leading terms of $V_{\mathrm{Yuk}}(x)$ are respectively given by the usual Coulombian term $\frac{2\beta}{x}$ and a constant $\frac{-2\beta}{\lambda}$. So we have

$$\lambda^2 V_{\mathrm{def}}(y) \simeq \left(\frac{g}{2y^2} + \frac{2\lambda\beta}{y} - 2\lambda\beta\right) + 0(y). \tag{13}$$

Note that for the particular case where $\beta \sim \frac{g}{\lambda}$, we find that $V_{\mathrm{def}}(x)$ is given by a deviation around $\frac{g}{2x^2}$. The behaviour $\beta \sim \frac{g}{\lambda}$ can be used to fix one of the two conditions for determining a 'particular solution' of the wave equation (11). This behaviour may be then interpreted as a natural physical boundary condition of the deformed system at $x = 0$. In this view, one also sees that the Calogero system is recovered from the DCS by taking a large wave length λ limit. For $x \gg \lambda$, the spectrum of the quantum system is mainly given by the usual Calogero one since in this range the interaction $\frac{2\beta}{x}\exp\left(-\frac{x}{\lambda}\right) \sim 0$.

Having these properties in mind, we turn now to establish integrability of the deformed model. The Schrödinger equation describing the quantum dynamics of these interacting particles reads as

$$\left(-\frac{1}{2}\sum_{i=1}^{2}\frac{\partial^2}{\partial x_i^2} + V(x_1, x_2)\right)\Psi(x_1, x_2) = E\Psi(x_1, x_2), \tag{14}$$

where Ψ is the total wavefunction, $V(x_1, x_2)$ is the total translation invariant potential given by

$$V(x) = \frac{\omega^2}{2}x^2 + \frac{g}{2x^2} + \frac{\sqrt{2}\beta}{x}\exp\left(-\frac{\sqrt{2}x}{\lambda}\right), \tag{15}$$

and E is the total energy eigenvalue depending on the potential moduli ω, g, β and λ.

To get the discrete spectrum $\{E_n, \Psi_n(x)\}$ of equations (52)–(59), we use translation invariance ($x_i \to x_i + cst$ implying $\Psi(x_1, x_2) = \Psi(x_1 - x_2)$) and take advantage on what we know about the Calogero solution for the wavefunction. We can thus decompose $\Psi_n(x)$ in three factors as follows:

$$\Psi_n(x) = x^{\varepsilon} \exp\left(-\frac{\omega}{2}x^2\right) \mathrm{F}_n(x), \tag{16}$$

where *a priori* ϵ is positive; but quantum effects require that it should be as $\epsilon > -\frac{1}{2}$, see [27] for details. Finding $\Psi(x)$ is then equivalent to (i) determining the ϵ parameter in terms of potential moduli and (ii) build the unknown function $\mathrm{F}(x)$ which, under physical requirements, should be constrained as

$$\int_0^{\infty} |\Psi(x)|^2 \,\mathrm{d}x = \int_0^{\infty} \mathrm{d}x\, \mathrm{e}^{-\omega x^2} |x^{\epsilon}\mathrm{F}|^2 < \infty. \tag{17}$$

Note that near infinity, property (10) shows that for $x \to \infty$ we should have the behaviour $\Psi_n(x) \sim \Psi_n^{(\mathrm{cal})}(x)$ implying in turns that $\mathrm{F}_n(x)$ should asymptotically behave like the Laguerre polynom $\mathrm{L}_n(x)$ of the Calogero solution. This feature means that in the formal expansion of $\mathrm{F}_n(x)$, the modes $b_{n,k}$ should be bounded as

$$|b_{n,k}| \leqslant \Lambda, \tag{18}$$

for some positive number Λ. With this physical requirement, we show in appendix A that we have $\lim_{k\to\infty} b_{n,k} = 0$.

Substituting $\Psi' = \frac{\mathrm{d}\Psi}{\mathrm{d}x}$, $\Psi' = \left[\mathrm{F}' + \left(\frac{\epsilon}{x} - \omega x\right)\mathrm{F}\right] x^{\varepsilon} \exp\left(-\frac{\omega}{2}x^2\right)$ and

$$\Psi'' = \left[\mathrm{F}'' + \left(\frac{2\epsilon}{x} - 2x\omega\right)\mathrm{F}' + \left(\frac{\epsilon(\epsilon-1)}{x^2} - (2\epsilon+1)\omega + \omega^2 x^2\right)\mathrm{F}\right] x^{\varepsilon}\, \mathrm{e}^{-\frac{\omega}{2}x^2} \tag{19}$$

in the Schrödinger equation, we get the following differential equation on F:

$$\mathrm{F}'' + 2\left(\frac{\epsilon}{x} - x\omega\right)\mathrm{F}' + \left(\frac{\epsilon^2 - \epsilon - g}{x^2} + 2E - (2\epsilon+1)\omega - \frac{\sqrt{8}\beta\, \mathrm{e}^{\frac{-x\sqrt{2}}{\lambda}}}{x}\right)\mathrm{F} = 0. \tag{20}$$

This is a second-order differential equation with non-constant coefficients which, as we know, its solutions is not a simple matter and requires specifying two 'boundary conditions'. To approach the solution of this differential equation, we shall proceed as follows.

(i) Use properties (9) to fix the first condition as

$$\left(\frac{\Psi_n(x)}{x^{\varepsilon}}\right)_{x=0} = \mathrm{F}_n(0) = \frac{1}{n!}\prod_{j=1}^{n}\left(j + \epsilon - \frac{1}{2}\right), \qquad n \geqslant 1 \tag{21}$$

and $\mathrm{F}_0(0) = 1$ for the ground state. This condition tells that, like for the Calogero case, the wavefunction has a node at the origin with degeneracy ϵ; $\mathrm{F}_n(0)$ is then just the value of the Laguerre polynom $L_n^{(\epsilon-\frac{1}{2})}(\omega x^2)$ at $x = 0$. Similarly, the second condition is given by

$$\left(\frac{\Psi_n(x)}{x^{\varepsilon}}\right)'_{x=0} = \mathrm{F}_n'(0) = \frac{\beta\sqrt{2}}{\epsilon}\mathrm{F}_n(0). \tag{22}$$

It will be determined later on; see equation (33). Note that

$$\frac{\mathrm{F}_n'(0)}{\mathrm{F}_n(0)} = \frac{\beta\sqrt{2}}{\epsilon} \tag{23}$$

is independent of the integer n.

(ii) Use the expansion method of the function $\mathrm{F}_n(x)$ to look for wave solutions given by formal series as described below.

2.2. *Expansion method*

First write $\mathrm{F}(x)$ in the form of the integral series

$$\mathrm{F}(x) = \sum_{k=0}^{\infty} b_k x^k, \tag{24}$$

where the modes $b_k = b_k(\epsilon, \omega, \beta, \lambda)$, $k \geqslant 2$, which depend on the coupling moduli, are now the new unknown quantities which have to be determined. The integer n is implemented by considering the expansion $\mathrm{F}_n(x) = \sum_{k=0}^{\infty} b_{n,k} x^k$ with $b_{n,0}$ and $b_{n,1}$ as specified above. Moreover, the normalization condition (17) which reads also as

$$\int_0^{\infty} |\Psi(x)|^2 \,\mathrm{d}x = \int_0^{\zeta} |\Psi(x)|^2 \,\mathrm{d}x + \int_{\zeta}^{\frac{1}{\zeta}} |\Psi(x)|^2 \,\mathrm{d}x + \int_{\frac{1}{\zeta}}^{\infty} |\Psi(x)|^2 \,\mathrm{d}x < \infty, \tag{25}$$

for a generic positive parameter ζ, requires that near the origin ($x \in [0, \zeta]$) the leading term of the integral $\int_0^{\zeta} \mathrm{e}^{-\omega x^2} |x^{\epsilon} \mathrm{F}|^2 \,\mathrm{d}x$ gives

$$\lim_{\zeta \to 0} \int_0^{\zeta} \mathrm{d}x \left(b_{n,0}^2 x^{2\epsilon} \right) = b_{n,0}^2 \lim_{\zeta \to 0} \left(\frac{\zeta^{2\epsilon+1}}{2\epsilon + 1} \right). \tag{26}$$

Positivity and finiteness imply that we should have $\epsilon > -\frac{1}{2}$.

To get the explicit expression of the new unknown factors b_k, we start from the differential equation (20). Then, putting the expansion of $\mathrm{F}(x)$ back into equation (20) and using the development of $\exp\left(\frac{-x\sqrt{2}}{\lambda}\right)$, namely $\sum_{m=0}^{\infty} \frac{(-)^m 2^{\frac{m}{2}}}{\lambda^m m!} x^m$, we get after term rearrangements the following algebraic equation:

$$\sum_{k=0}^{\infty} B_k x^{k-2} = 0, \tag{27}$$

where the two first B_k modes are given by

$$\begin{aligned} B_0 &= [\epsilon(\epsilon - 1) - g] b_0, \\ B_1 &= (2\epsilon + \epsilon(\epsilon - 1) - g) b_1 - 2\beta\sqrt{2} b_0. \end{aligned} \tag{28}$$

The remaining others ($k \geqslant 2$) read collectively like

$$B_k = [k(2\epsilon + k - 1) + \epsilon(\epsilon - 1) - g] b_k - D_k, \tag{29}$$

with D_k s given by

$$D_k = 2\beta \sum_{m=0}^{k-1} \frac{(-)^{k-m-1} 2^{\frac{k-m}{2}}}{(k - m - 1)! \lambda^{k-m-1}} b_m + [(2k + 2\epsilon - 3)\omega - 2E] b_{k-2}. \tag{30}$$

The solution of equation (27) for arbitrary x is obtained by requiring the vanishing of the B_k coefficients which give in turn the following recurrent constraint equations on the b_k modes. From equations (28)–(29), one learns the followings.

(a) Equations (28) have no dependence in the energy E:

$$0 = [\epsilon(\epsilon - 1) - g] b_0, \tag{31}$$

$$0 = -\beta\sqrt{2} b_0 + \left[\epsilon + \frac{\epsilon(\epsilon - 1) - g}{2} \right] b_1. \tag{32}$$

They permit to fix the second condition $\mathrm{F}'(0) = b_1$. Indeed solving equations (31)–(32), we find

$$g = \epsilon(\epsilon - 1), \qquad b_1 = \frac{\beta\sqrt{2}}{\epsilon} b_0. \tag{33}$$

(b) Equation (29) has a linear dependence in E,

$$0 = \left[E - \left(\epsilon + \frac{1}{2}\right)\omega + \frac{2\beta}{\lambda}\right] b_0 - \beta\sqrt{2} b_1 + [(4\epsilon + 2) + \epsilon(\epsilon - 1) - g]\frac{b_2}{2}, \tag{34}$$

together with ($k \geqslant 3$):

$$0 = -\beta \sum_{m=0}^{k-3} \frac{(-)^{k-m-1} 2^{\frac{k-m}{2}}}{(k - m - 1)!\lambda^{k-m-1}} b_m + \left[E - \left(k + \epsilon - \frac{3}{2}\right)\omega + \frac{2\beta}{\lambda}\right] b_{k-2}$$
$$- \beta\sqrt{2} b_{k-1} + \frac{1}{2}[k(2\epsilon + k - 1) + \epsilon(\epsilon - 1) - g] b_k. \tag{35}$$

This is an infinite system and we need to specify some regularization method to deal with it. We shall first work heuristically as if the above system of equations is finite, but very large. Later on we show how to deal with the infinite dimensionality.

To get DCS-quantum spectrum, we have to determine E and the $\{b_k, k \in \mathbb{N}\}$ modes in terms of the moduli,

$$E = E(\epsilon, \beta, \lambda; n) \qquad b_k = b_k(\epsilon, \beta, \lambda; n), \tag{36}$$

where emergence of the quantum number n will be discussed later on.

3. DCS spectrum

To deal with the above infinite-dimensional system equations (34), (35), it is interesting to rewrite it in a formal infinite matrix form. The method is as follows.

(1) Use specific properties of equations (34), (35) to factorize them into the product of two blocks, one involving the parameter E only, and the second sector involves the modes b_k. This is achieved by using matrix formalism as shown below:

$$\sum_{j=1}^{\infty} M_{kj} c_j = 0, \qquad k = 1, \ldots, \tag{37}$$

where we have set $b_k = c_{k+1}$ and M_{kj} is a function of energy, $M_{kj} = M_{kj}(E)$, with entries

$$M_{1j} = \left[E - \left(\epsilon + \frac{1}{2}\right)\omega + \frac{2\beta}{\lambda}\right]\delta_{1j} - \beta\sqrt{2}\delta_{2j} + \frac{(4\epsilon + 2) + \epsilon(\epsilon - 1) - g}{2}\delta_{3j} \tag{38}$$

and for $k \geqslant 3$,

$$M_{k-1,j} = -\frac{(-)^{k-j} 2^{\frac{k+1-j}{2}} \beta}{(k - j)!\lambda^{k-j}}\theta(k - 2 - j) + \left[E - \left(k + \epsilon - \frac{3}{2}\right)\omega + \frac{2\beta}{\lambda}\right]\delta_{j,k-1}$$
$$- \beta\sqrt{2}\delta_{j,k} + \frac{1}{2}[k(2\epsilon + k - 1) + \epsilon(\epsilon - 1) - g]\delta_{j,k+1}, \tag{39}$$

where $\theta(u) = 1$ for $u \geqslant 0$ and $\theta(u) = 0$ for $u < 0$.

(2) Compute the explicit expression of the energy E in terms of the coupling constant moduli by using M invariant. As we will show, E will be determined by computing $\det M$.
(3) Once we get the explicit form of E, we insert it in (37) to get the expression of the b_k modes.

3.1. Determining E

The determination of the energy eigenvalues of the DCS model is obtained by first computing the determinant of the matrix M and then look for its zeros. The last property is due to the fact that the solutions of the finite-dimensional regularization of equation (37) require

$$\det M = 0. \tag{40}$$

As one notes, the explicit form of $\det M$ is not an obvious matter even by using the finite-dimensional regularization of the DCS matrix (38)–(39). In appendix B, we show rigourously how this can be done; below we summarize the main steps of the computation of E_n in order to not loose the logic of calculation.

To that purpose, we begin by giving the leading entries of the infinite matrix M; these are useful to fix the ideas and to learn directly some specific properties of this M matrix. We have

$$M = \begin{pmatrix} e & -\sqrt{2}\beta & 2\left(\epsilon+\frac{1}{2}\right) & 0 & 0 & . & . \\ M_{21} & e-\omega & -\sqrt{2}\beta & 3\left(\epsilon+\frac{2}{2}\right) & 0 & . & . \\ M_{31} & M_{21} & e-2\omega & -\sqrt{2}\beta & 4\left(\epsilon+\frac{3}{2}\right) & . & . \\ M_{41} & M_{31} & M_{21} & e-3\omega & -\sqrt{2}\beta & . & . \\ M_{51} & M_{41} & M_{31} & M_{21} & e-4\omega & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \end{pmatrix}, \tag{41}$$

with

$$\begin{aligned} M_{kk} &= e-(k-1)\omega, && k \geqslant 1 \\ M_{k1} &= \frac{2\beta(-\sqrt{2})^{k-1}}{k!\lambda^k}, && k \geqslant 2, \\ e &= E - \left(\epsilon+\frac{1}{2}\right)\omega + \frac{2\beta}{\lambda}. \end{aligned} \tag{42}$$

From these relations, we learn the two following properties. First, if one succeeds to determine the last relation of above equations, we see that the energy is given by $E = e + \left(\epsilon+\frac{1}{2}\right)\omega - \frac{2\beta}{\lambda}$. Second, the matrix entries M_{ij} have the remarkable property,

$$\frac{M_{k,1}}{M_{k+1,1}} = -\frac{(k+1)\lambda}{2}\sqrt{2}, \tag{43}$$

which play a crucial role in the computation of the determinant; see also equation (A.7).

To compute $\det M$ and then solve the condition $\det M = 0$, we use a set of specific properties of M which are established in appendix B. Using the results derived there, the steps leading to E_n can be summarized as follows.

(i) Think about the infinite-dimensional matrix M as a large $q \times q$ matrix M_q. Once we end the study of the remarkable properties of M_q and determine $\det M_q$, we then take the limit for the infinite matrix $M_\infty \equiv M$.

(ii) To compute $\det M_q$, we shall proceed in an indirect way having seen that we cannot do it directly. We use p successive similarity transformations to put M_q into different, but equivalent, forms

$$M_q^{(p+1)} = U_p M_q^{(p)} U_p^{-1}, \qquad M_q^{(1)} = M_q \qquad p = 1, 2, 3, \ldots, \tag{44}$$

where U_p s are similarity transformations. In appendix B, we build the $M_q^{(p)}$ family for $p = 1, 2, 3, \ldots,$ and give the result for generic p. The generic matrix $M_q^{(p)}$ has the properties

- $\det M_q = \det M_q^{(p)}$, for any positive integer p;
- the matrix $M_q^{(p)}$ has the following matrix block structure:

$$M_q^{(p)} = \begin{pmatrix} T_{p\times p} & R_{p\times(q-p)} \\ L_{(q-p)\times p} & K_{(q-p)\times(q-p)} \end{pmatrix}. \tag{45}$$

Here T and K are $p \times p$ and $(q-p)\times(q-p)$ square matrices, respectively, and L and R are rectangular ones. It happens that only the properties of T and L are important for the determination of $\det M_q^{(p)}$. Using the results of appendix B, it is shown that $T_{p\times p}$ is a $p \times p$ triangular matrix with determinant

$$\det T = \prod_{n=1}^{p} \Delta_n, \tag{46}$$

where Δ_n is given by

$$\Delta_n = \bigg[e_{n-1} - (n-1)\omega \\ - \frac{(n+1)\alpha}{\mu}\left(\frac{1}{2!} + \frac{1-n}{3!} + \cdots + \frac{(1-n)(2-n)\cdots(-2)}{n!}\right)\bigg],$$

and where we have set $\mu = \frac{-\sqrt{2}}{\lambda}$ and $\alpha = -\sqrt{2}\beta\mu^2$. Note in passing that the matrix $L_{(q-p)\times p}$ is an almost vanishing matrix. As shown in appendix B, equation (A.15), it happens that the last line and the last column of L which are non-zero are given below:

$$(L_{ij}) = \left(M_q^{(p)}\right)_{p+i,j} = \begin{cases} 0, & i \neq q-p \text{ and } j \neq p. \\ m_p \neq 0 & \text{otherwise} \quad (i = q-p \text{ or } j = p). \end{cases} \tag{47}$$

Here $1 \leqslant i \leqslant q-p$ and $1 \leqslant j \leqslant p$ and m_p are some values whose explicit expressions are not needed for the computation of $\det M^{(p)}$. Note that for the computation of the determinant of $M_q^{(p)}$, it is interesting to think about $M_q^{(p)}$ as shown below:

$$M_q^{(p)} = \begin{pmatrix} \widetilde{T}_{(p-1)\times(p-1)} & \widetilde{R}_{(p-1)\times(q-p+1)} \\ \widetilde{L}_{(q-p+1)\times(p-1)} & \widetilde{K}_{(q-p+1)\times(q-p+1)} \end{pmatrix},$$

where $\widetilde{T}_{(p-1)\times(p-1)}$ is the $(p-1)\times(p-1)$ matrix restriction of $T_{p\times p}$ and where now only the last line of $\widetilde{L}_{(q-p+1)\times(p-1)}$ is non-zero. We leave the technical details to the appendix, but keep in mind that $\det M^{(p)} = \det \widetilde{T} . \det \widetilde{K}$ together with the discussion following equation (43).

(iii) Using the above-mentioned results, equations (37) can be put in the block form

$$\begin{pmatrix} \widetilde{T}_{(p-1)\times(p-1)} & \widetilde{R}_{(p-1)\times(q-p+1)} \\ \widetilde{L}_{(q-p+1)\times(p-1)} & \widetilde{K}_{(q-p+1)\times(q-p+1)} \end{pmatrix}_{ij} \begin{pmatrix} c_j^{(p-1)} \\ c_j^{(q-p+1)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{48}$$

where $c_j^{(p-1)}$ are the new variables related to c_j by the pth similarity transformation U_p.

(iv) Use the fact that in the infinite-dimensional limit, the matrix $\widetilde{L}_{(q-p+1)\times(p-1)}$ is basically zero equation (A.15), and moreover the ability to take the integer p as large as possible to end with the result; see also equation (A.16) for rigorous derivation

$$\det M = \left(\lim_{p\longrightarrow\infty} \det T_{p\times p}\right) \det K = 0, \tag{49}$$

where we have dropped out the twild symbol ($\sim$). Though heuristic, one can extract from this equation important information; it has infinitely many solutions, in particular those given by taking

$$\lim_{p\longrightarrow\infty} \det T_{p\times p} = 0. \tag{50}$$

Obviously $\det M = 0$ can be solved in various ways. Besides $\det T = 0$, one may also have $\det K = 0$; it is also a possible solution of equation (49), but we have no way to compute it explicitly.

Since T is a triangular matrix and seen that diagonal terms $T_{nn} = M_{nn}^{(p)}$, we have

$$\prod_{n=1}^{\infty}\left[e_{n-1} - (n-1)\omega - \frac{(n+1)\alpha}{\mu}\left(\frac{1}{2!} + \frac{1-n}{3!} + \cdots + \frac{(1-n)(2-n)\cdots(-2)}{n!}\right)\right] = 0, \tag{51}$$

where now

$$e_{n-1} = E_{n-1} - \left(\epsilon + \frac{1}{2}\right)\omega + \frac{2\beta}{\lambda}, \qquad n \geqslant 1. \tag{52}$$

There are infinitely many solutions of equation (51), but because of the symmetry $x \leftrightarrow (-x)$ of the classical Hamiltonian and positivity of Yukawa interaction, only half of these solutions are physical. Before giving more details, let us first derive the explicit expression of E_n and then come back to the physical spectrum. At a generic integer level $n \neq 0$, we have then

$$e_{n-1} - (n-1)\omega - \frac{(n+1)\alpha}{\mu}\left(\frac{1}{2!} + \frac{1-n}{3!} + \cdots + \frac{(1-n)(2-n)\cdots(-2)}{n!}\right) = 0. \tag{53}$$

Using equations (52) and shifting $n \to n+1$, we find for $n \geqslant 0$:

$$E_n = \left(n + \epsilon + \frac{1}{2}\right)\omega - \frac{2\beta}{\lambda} + (n+2)\left[\frac{1}{2!} - \frac{n}{3!} + \cdots + (-1)^{n-1}\frac{n(n-1)(n-2)\cdots 2}{(n+1)!}\right]\frac{2\beta}{\lambda}. \tag{54}$$

Upon multiplying and dividing the second term by the number $(n+1)$, then adding and subtracting the quantity $\frac{1}{(n+1)}\left[C_{n+2}^0 - C_{n+2}^1 + (-1)^{n+2}C_{n+2}^{n+2}\right]$, E_ns can be rewritten as follows:

$$E_n = \left(n + \epsilon + \frac{1}{2}\right)\omega - \frac{2\beta}{\lambda}\left(1 - \frac{1}{(n+1)}\sum_{k=0}^{n+2}(-1)^k C_{n+2}^k\right) - \frac{2\beta}{(n+1)\lambda}\left[C_{n+2}^0 - C_{n+2}^1 + (-1)^{n+2}C_{n+2}^{n+2}\right], \tag{55}$$

where we have set $C_n^k = \frac{n!}{k!(n-k)!}$. Since the sum $\sum_{k=0}^{n+2}(-1)^k C_{n+2}^k$ adds to zero, we end with the following result:

$$E_n = \left(n + \epsilon + \frac{1}{2}\right)\omega + \frac{2\beta(-1)^{n+1}}{\lambda(n+1)}, \tag{56}$$

which is formula (2) given in the introduction. This relation deserves some comments.

(i) Physical energies correspond to truncate the above spectrum by $\mathbb{Z}_2$ symmetry of the classical Hamiltonian. This means that one should keep either E_{2n} or E_{2n+1}; but the positivity of the Yukawa potential shows that the physical result should be as

$$E_{2n+1} = \left(2n + \epsilon + \frac{3}{2}\right)\omega + \frac{\beta}{\lambda(n+1)}. \tag{57}$$

(ii) In the particular case, where $\beta = \frac{g}{\lambda} = \frac{\epsilon(\epsilon-1)}{\lambda}$, the energy E_{2n+1} becomes

$$E_{2n+1}\left(\beta = \frac{g}{\lambda}\right) = \left(2n + \epsilon + \frac{3}{2}\right)\omega + \frac{1}{(n+1)}\frac{\epsilon(\epsilon-1)}{\lambda^2}. \tag{58}$$

In this case, for $\epsilon = 0$ or $\epsilon = 1$, one has the spectrum of the harmonic oscillator.

(iii) The third comment deals with the limits $\lambda \to \infty$ and $\lambda \to 0$. These limits depend on the coupling constant β. In the limit $\lambda \to \infty$, the coupling constant β can be taken as an independent modulus of λ (or in general as $\beta = \beta_0 + \frac{g}{\lambda}$) and so one falls in the usual Calogero spectrum. However for $\lambda \to 0$, the convergence of physical energy and consistency requires

$$\frac{\beta}{\lambda} \sim \omega. \tag{59}$$

Combining these comments, we end with the result that an exact and finite solution of the spectrum of the deformed Calogero model by the Yukawa-like potential requires a condition like (59) which can be taken as $\frac{\beta}{\lambda} = \omega$. In this case the exact energy is given by equation (3).

3.2. Determining eigenfunctions

Knowing the explicit expression of the spectrum E_n, substitute it in the underlying equations (27) which we rewrite as

$$\sum_{k=0}^{\infty} B_{n,k} x^{k-2} = 0, \tag{60}$$

which lead to similar relations as in equations (34)–(35),

$$b_{n,k} = \frac{-2}{k(2\epsilon + k - 1)}\left[\sum_{l=0}^{k-1} \frac{\beta(-\sqrt{2})^{k-l} b_{n,l}}{(k-l-1)!\lambda^{k-l-1}} + \left((n-k+2)\omega + \frac{(-1)^{n+1}2\beta}{(n+1)\lambda}\right) b_{n,k-2}\right]. \tag{61}$$

Using boundary conditions and equation (61), we can determine by iteration all the modes $b_{n,k}$; all one needs are the values of the two leading modes of $b_{n,0}$ and $b_{n,1}$. These are given by $b_{0,0} = 1$, $b_{0,1} = \frac{\beta\sqrt{2}}{\epsilon}$ for the ground state Ψ_0 and $b_{n,0} = \frac{1}{n!}\prod_{j=1}^{n}\left(\epsilon - \frac{1}{2} + j\right)$, $b_{n,1} = \frac{\beta\sqrt{2}}{\epsilon} b_{n,0}$ for Ψ_n with $n \geqslant 1$. The physical wavefunctions Υ_n have energies $\mathcal{E}_n$ equation (3) and are obtained by $\mathbb{Z}_2$ truncation of the space of the Ψ_n.

4. Conclusion

In this work, we have computed the explicit expression of the discrete spectrum of a system obtained by the deformation of Calogero potential by a Yukawa-like coupling. This extra interaction models strong coupling between particles and constitutes a step towards getting more insight beyond the exact Calogero analysis.

Besides the explicit expression of the discrete energies $\mathcal{E}_n$ and wavefunctions Υ_n which constitutes the basic purpose of this study, our analysis offers another way to define integrability by using boundary conditions on the deformation potential. In the example we have developed in this paper, the deformed potential

$$V_{\text{def}}(x) = \frac{g}{2x^2} + V_{\text{Yuk}}(x), \qquad 0 < x < \infty,$$

behaves like $\frac{g}{2x^2}$ for both the UV region $x \to 0$ and infrared $x \to \infty$,

$$V_{\text{def}}(x) \underset{x\to 0}{\longrightarrow} \frac{g}{2x^2}, \qquad V_{\text{def}}(x) \underset{x\to\infty}{\longrightarrow} \frac{g}{2x^2}. \tag{62}$$

These boundary conditions have been used to fix two modes of the expansion of the wavefunction $\Psi_n(x) = \sum_{k=0}^{\infty} b_{n,k}x^k$. These modes concern b_0 and b_∞ and are required to take the values $b_{n,0} = \frac{1}{n!}\prod_{j=1}^{n}\left(j+\epsilon-\frac{1}{2}\right)$, $n \geqslant 1$ $(b_{0,0} = 1)$ and $b_{n,\infty} = 0$ (see equation (A.1) of appendix A). They ensure the convergence of the integral of the total probability density, $\int_0^\infty \mathrm{d}x\,|\Psi_n(x)|^2$, and agree with the Calogero limit. Moreover, having seen that the knowledge of $\Psi_n(x)$ is given by solving the second-order differential equation (11), we have two arbitrary degrees of freedom which can be used to fix two parameters of the DCS model. It happens that the degrees of freedom in question are b_0 and b_1. b_0 has been already fixed by the Calogero limit $(x \to 0)$ and $b_1 = \frac{\beta\sqrt{2}}{\lambda}b_0$. We think that it would be interesting to apply this method to other deformations and other (quasi-) integrable models to get the explicit expression of the energies as we have done above.

Acknowledgment

This research work is supported by the program Protars III D12/25, CNRST.

Appendix.

We give two appendices: one dealing with the formal expansion of the factor $\mathrm{F}_n(x)$ of the factorization (16) and the second with the similarity transformations allowing us to put the matrix M into equation (45).

Appendix A.1

Following splitting (16) and the physical hypothesis on the modes $b_{n,k}$ of the $\mathrm{F}_n(x) = \frac{\Psi(x)}{x^{\varepsilon}}$ (18), we would like to establish the following.

Proposition 1. *Given the series* $\mathrm{F}_n(x) = \sum_{k=0}^{\infty} b_{n,k}x^k$ *satisfying the second-order equation (20); then if* $b_{n,k}$ *s satisfy equation (18) for any finite integer* n*, we have the result*

$$\lim_{k\longrightarrow\infty} b_{n,k} = 0. \tag{A.1}$$

To establish this statement, we start from the following relation expressing $b_{n,k}$ in terms of the other $b_{n,j}$ s, $0 \leqslant j < k$:

$$b_{n,k} = \frac{2}{k(2\epsilon+k-1)}\left\{\beta\sum_{m=0}^{k-1}\frac{(-)^{k-m-1}2^{\frac{k-m}{2}}}{(k-m-1)!\lambda^{k-m-1}}b_{n,m} + \left[\left(k+\epsilon-\frac{3}{2}\right)\omega - E\right]b_{n,k-2}\right\}. \tag{A.2}$$

This relation is obtained by combining equations (20) and (18). Then using the bound $|b_{n,m}| \leqslant \Lambda$, we can compute the behaviour of $b_{n,k}$ for large k. All one has to do is to replace the first term by

$$\begin{aligned} \left|\sum_{m=0}^{k-1} \frac{(-)^{k-m-1} 2^{\frac{k-m}{2}}}{(k-m-1)!\lambda^{k-m-1}} b_{n,m}\right| &\leqslant \Lambda \sum_{m=0}^{k-1} \frac{2^{\frac{k-m}{2}}}{(k-m-1)!\lambda^{k-m-1}}, \\ \left|\left[\left(k+\epsilon-\frac{3}{2}\right)\omega - E\right] b_{n,k-2}\right| &\leqslant \Lambda\left[\left(k+\epsilon-\frac{3}{2}\right)\omega - E\right]. \end{aligned} \tag{A.3}$$

By taking the limit $k \to \infty$ and using the identity $\lim_{k\longrightarrow\infty} \sum_{m=0}^{k-1} \frac{2^{\frac{k-m}{2}}}{(k-m-1)!\lambda^{k-m-1}} = \sqrt{2}\exp\left(\frac{\sqrt{2}}{\lambda}\right)$, we get

$$\begin{aligned} \lim_{k\longrightarrow\infty} |b_{n,k}| \leqslant \lim_{k\longrightarrow\infty} \frac{2\Lambda}{k(2\epsilon+k-1)} &\left\{\left|\sum_{m=0}^{k-1} \frac{(-)^{k-m-1}\beta}{(k-m-1)!\lambda^{k-m-1}} 2^{\frac{k-m}{2}}\right|\right. \\ &\left.+ \left|\left(k+\epsilon-\frac{3}{2}\right)\omega - E\right|\right\}, \end{aligned} \tag{A.4}$$

implying in turn

$$\lim_{k\longrightarrow\infty} |b_{n,k}| \leqslant \Lambda\sqrt{2}\exp\left(\frac{\sqrt{2}}{\lambda}\right) \lim_{k\longrightarrow\infty} \frac{2\beta}{k(2\epsilon+k-1)} = 0, \tag{A.5}$$

in agreement with the result of the proposition.

Appendix A.2

Here we would like to establish equation (45). To get this result, we shall use an explicit method by making successive similarity transformations. The general result is summarized in the following.

Proposition 2. *Given the matrix M of equations (38)–(39) with its remarkable property $M_{l+r,j+r} = M_{l,j} \neq 0$, $j < l$*

(1) *One can build a series of equivalent matrices $M^{(j)}$ $j = 1, 2, \ldots, m, \ldots$ related to M by similarity transformations as shown below ($M = M^{(1)}$):*

$$\begin{aligned} &M^{(2)} = U_1 M^{(1)} U_1^{-1}, \qquad M^{(3)} = U_2 M^{(2)} U_2^{-1}, \qquad M^{(j+1)} = U_j M^{(j)} U_j^{-1} \\ &\qquad\ldots \\ &M^{(m+1)} = \left(\prod_{j=1}^{m} U_j^{-1}\right)^{-1} M \left(\prod_{j=1}^{m} U_j^{-1}\right), \qquad \ldots, \end{aligned} \tag{A.6}$$

where U_j s are invertible matrices and where all '$\det M^{(j)}$' are same as '$\det M$'.

(2) *The ratio $\xi_l = \frac{M_{l,1}}{M_{l+1,1}}$ is invariant under the transformations (A.6)*

$$\frac{M^{(m)}_{l,m}}{M^{(m)}_{l+1,m}} = -\frac{(l+1)\lambda}{2}\sqrt{2}, \qquad m = 1, 2, \ldots. \tag{A.7}$$

This result may be viewed directly in equation (41).

(3) The entries of the matrix $M^{(j)}$ for $l \leqslant k$ with $j \geqslant 2$ are given by

$$M_{kl}^{(j)} = \frac{(1-l)(2-l)\cdots(j-1-l)}{(k+j-l)!}\alpha\mu^{k-l-1}; \qquad l < k$$
$$M_{kk}^{(j)} = e - (k-1)\omega - \frac{k+1}{\mu}\left(\frac{1}{2!} + \frac{1-k}{3!} + \cdots + \frac{(1-k)(2-k)\cdots(j-2-k)}{j!}\right)\alpha, \tag{A.8}$$

where we have set $\mu = \frac{-\sqrt{2}}{\lambda}$, $\alpha = -\sqrt{2}\beta\mu^2$.

To build the $M^{(j)}$ s, we proceed using an explicit manner and do things by iteration. The matrix $M^{(2)}$ is given by

$$M_{1,j}^{(2)} = M_{1j}^{(1)}, \qquad M_{l,j}^{(2)} = M_{lj}^{(1)} - \xi_l M_{l+1,j}^{(1)}, \qquad l \geqslant 2. \tag{A.9}$$

The U_1 similarity transformation has then the property of fixing $M_{11}^{(1)}$ and annihilating the remaining entries of the first column,

$$M_{11}^{(2)} = M_{11}^{(1)} \qquad M_{l,1}^{(2)} = 0, \qquad l \geqslant 2. \tag{A.10}$$

Explicit computation gives, amongst others,

$$\begin{cases} M_{kl}^{(2)} = M_{kl}^{(1)} - \frac{k+1}{\mu} M_{k+1,l}^{(1)} = \frac{1-l}{(k+2-l)!}\alpha\mu^{k-l-1}; \qquad l < k \\ M_{kk}^{(2)} = M_{kk}^{(1)} - \frac{k+1}{\mu} M_{k+1,k}^{(1)} = e - (k-1)\omega - \frac{k+1}{2!}\frac{\alpha}{\mu}, \end{cases} \tag{A.11}$$

where, except the first element, one recognizes that the first column ($l = 1$) is zero. The same method is used to build $M^{(3)}$, we have

$$\begin{aligned} M_{l,j}^{(3)} &= M_{l,j}^{(2)}, & l &= 1, 2 \\ M_{l,j}^{(3)} &= M_{lj}^{(2)} - \xi_l M_{l+1,j}^{(2)}, & l &\geqslant 3. \end{aligned} \tag{A.12}$$

More generally, we have

$$\begin{aligned} M_{l,j}^{(m+1)} &= M_{l,j}^{(m)}, & l &= 1, \ldots, m \\ M_{l,j}^{(m+1)} &= M_{lj}^{(m)} - \xi_l M_{l+1,j}^{(m)}, & l &\geqslant m+1. \end{aligned} \tag{A.13}$$

We also have

$$\begin{cases} M_{kl}^{(n)} = \frac{(1-l)(2-l)\cdots(n-1-l)}{(k+n-l)!}\alpha\mu^{k-l-1}; \qquad l < k \\ M_{kk}^{(n)} = e - (k-1)\omega - \frac{k+1}{\mu}\left(\frac{1}{2!} + \frac{1-k}{3!} + \cdots + \frac{(1-k)(2-k)\cdots(n-2-k)}{n!}\right)\alpha, \end{cases}$$

where one sees that the entries $M_{kl}^{(n)}$ vanish for $l = 1, 2, \ldots, n-1$ and $l < k$. Note also that one can write $M_{kl}^{(n)}$ in matrix blocks as

$$\begin{pmatrix} T & R \\ L & K \end{pmatrix}, \tag{A.14}$$

where T is a triangular matrix with diagonal elements $T_{jj} = M_{jj}^{(n)}$.

Corollary 3. *Given the matrix $M^{(n)}$ with entries as in equations (A.8), we have for finite l,*

$$\lim_{k\longrightarrow\infty} M_{kl}^{(n)} = 0, \tag{A.15}$$

showing that one may approximate L with zero matrix. So, we roughly have

$$\det M^{(n)} \simeq \det T \det K. \tag{A.16}$$

Equation (A.15) may be obtained by computing

$$M_{kl}^{(m)} = (1-l)(2-l)\cdots(m-1-l)\alpha\frac{\mu^{k-l-1}}{(k+m-l)!}. \tag{A.17}$$

The term $\frac{\mu^{k-l-1}}{(k+m-l)!}$ is equivalent to the general term of the convergent positive series $\exp(\mu)$, and consequently

$$\lim_{k\longrightarrow\infty}\left(\frac{\mu^{k-l-1}}{(k+m-l)!}\right) = 0, \tag{A.18}$$

thus $\lim_{k\to\infty} M_{kl}^{(m)} = 0$.

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